DIVERSITY READING LIST PRESENTS:

PEOPLE AND PROOFS

A DRL READING GROUP BLUEPRINT BY FENNER STANLEY TANSWELL
LEVEL: INTERMEDIATE-HARD

INTRODUCTION

This blueprint is about the role that people play in mathematics and its practices. Traditional philosophy of mathematics tends to idealise away from and ignore the human contexts, cultures, and practices that shape and underlie it. However, despite its abstract subject matter, mathematics is a social human discipline involving collaborations, communication, subjective evaluative judgements, power dynamics, norms, fallibility, and disagreements. The aim of this blueprint is to look at works that engage with these ways in which social features of mathematical practice affect the mathematics that is produced, who gets to produce it, and how it is evaluated.

A central theme of the blueprint will be about proofs and knowledge in mathematics. We will look at how the traditional notion of proof and its link to absolute certainty is challenged by practices involving testimony, probabilistic reasoning, large-scale and online collaboration, diagrams, and computer proofs. To engage with these topics, this blueprint contains a selection of readings that include works by philosophers, mathematicians, historians, social scientists, and data scientists. This emphasises the point that multiple perspectives and approaches are valuable in addressing philosophical issues in mathematics. While several of the papers do mathematical content, some of it a bit tricky, I have attempted to make this blueprint accessible to interested participants without a mathematical background. The mathematical content that there is can mostly be skimmed over or skipped altogether without losing too much of the spirit of the papers.

Each week contains a main reading and a secondary reading or other resource. These have been paired to complement one another, but the secondary resource can be set aside for a shorter discussion.

CATEGORIES

- Epistemology of Mathematics
- Mathematical Practice
- Mathematical Methodology
- Social Epistemology
- Sociology of Science

AVAILABLE ONLINE AT:

https://diversityreadinglist.org/blueprint/people-and-proofs

1. What is the role of philosophy in mathematics?

CHENG, EUGENIA. *MATHEMATICS, MORALLY*

2004, [conference talk], Cambridge University Society for the Philosophy of Mathematics. **Difficulty:** Intermediate

ABSTRACT:

A source of tension between Philosophers of Mathematics and Mathematicians is the fact that each group feels ignored by the other; daily mathematical practice seems barely affected by the questions the Philosophers are considering. In this talk I will describe an issue that does have an impact on mathematical practice, and a philosophical stance on mathematics that is detectable in the work of practising mathematicians. No doubt controversially, I will call this issue 'morality', but the term is not of my coining: there are mathematicians across the world who use the word 'morally' to great effect in private, and I propose that there should be a public theory of what they mean by this. The issue arises because proofs, despite being revered as the backbone of mathematical truth, often contribute very little to a mathematician's understanding. 'Moral' considerations, however, contribute a great deal. I will first describe what these 'moral' considerations might be, and why mathematicians have appropriated the word 'morality' for this notion. However, not all mathematicians are concerned with such notions, and I will give a characterisation of 'moralist' mathematics and 'moralist' mathematicians, and discuss the development of 'morality' in individuals and in mathematics as a whole. Finally, I will propose a theory for standardising or universalising a system of mathematical morality, and discuss how this might help in the development of good mathematics.

COMMENT:

Cheng is a mathematician working in Category Theory. In this article she complains about traditional philosophy of mathematics that it has no bearing on real mathematics. Instead, she proposes a system of "mathematical morality" about the normative intuitions mathematicians have about how it ought to be.

DISCUSSION QUESTIONS:

1. Cheng complains about philosophy of mathematics having no relevance to mathematical practice, and even gives a short dialogue featuring an obstinate philosopher. What do you think the impact of philosophy of mathematics on mathematics should be?
2. What is Cheng's notion of "mathematical moral truth"? Do you think it picks out a robust phenomenon?
3. Cheng says morality is about "how mathematics ought to behave". What kind of normativity do you think she has in mind?
4. What is the connection between expertise and feelings of mathematical morality?
5. Should/would a mathematician on a desert island do mathematics differently?

TAO, TERENCE. *WHAT IS GOOD MATHEMATICS?*

2007, *Bulletin of the American Mathematical Society*, 44(4): 623-634. **Difficulty:** Intermediate

Fragment: Section 1, pp 623-626.

ABSTRACT:

Some personal thoughts and opinions on what "good quality mathematics" is and whether one should try to define this term rigorously. As a case study, the story of Szemer´edi's theorem is presented.

COMMENT:

Tao is a mathematician who has written extensively about mathematics as a discipline. In this piece he considers what counts as "good mathematics". The opening section that I've recommended has a long list of possible meanings of "good mathematics" and considers what this plurality means for mathematics. (The remainder details the history of Szemerédi's theorem, and argues that good mathematics also involves contributing to a great story of mathematics. However, it gets a bit technical, so only look into it if you're particularly interested in the details of the case.)

DISCUSSION QUESTIONS:

6. Which of the listed qualities of good mathematics would benefit most from philosophical analysis?
7. Are some qualities of good mathematics more important than others?
8. Do you think mathematicians would agree on how to apply the various qualities? For example, would they agree on what counts as: rigorous maths? good pedagogy? mathematical beauty? good taste?
9. Do you agree with Tao that the standards of good mathematics in a field should be constantly debated and updated? Or do there exist eternal standards of good mathematics?

HAMAMI, YACIN AND REBECCA LEA MORRIS. *PHILOSOPHY OF MATHEMATICAL PRACTICE: A PRIMER FOR MATHEMATICS EDUCATORS*

2020, *ZDM*, 52(6): 1113-1126. **Difficulty:** Easy-Intermediate

ABSTRACT:

In recent years, philosophical work directly concerned with the practice of mathematics has intensified, giving rise to a movement known as the philosophy of mathematical practice. In this paper we offer a survey of this movement aimed at mathematics educators. We first describe the core questions philosophers of mathematical practice investigate as well as the philosophical methods they use to tackle them. We then provide a selective overview of work in the philosophy of mathematical practice covering topics including the distinction between formal and informal proofs, visualization and artefacts, mathematical explanation and understanding, value judgments, and mathematical design. We conclude with some remarks on the potential connections between the philosophy of mathematical practice and mathematics education.

COMMENT:

While this paper by Hamami & Morris is not a necessary reading, it provides a fairly broad overview of the practical turn in mathematics. Since it was aimed at mathematics educators, it is a very accessible piece, and provides useful directions to further reading beyond what is included in this blueprint.

2. Proof and Fallibility

DE TOFFOLI, SILVIA. *GROUNDWORK FOR A FALLIBILIST ACCOUNT OF MATHEMATICS*

2021, *The Philosophical Quarterly*, 71(4). **Difficulty:** Intermediate-Advanced

ABSTRACT:

According to the received view, genuine mathematical justification derives from proofs. In this article, I challenge this view. First, I sketch a notion of proof that cannot be reduced to deduction from the axioms but rather is tailored to human agents. Secondly, I identify a tension between the received view and mathematical practice. In some cases, cognitively diligent, well-functioning mathematicians go wrong. In these cases, it is plausible to think that proof sets the bar for justification too high. I then propose a fallibilist account of mathematical justification. I show that the main function of mathematical justification is to guarantee that the mathematical community can correct the errors that inevitably arise from our fallible practices.

COMMENT:

De Toffoli makes a strong case for the importance of mathematical practice in addressing important issues about mathematics. In this paper, she looks at proof and justification, with an emphasis on the fact that mathematicians are fallible. With this in mind, she argues that there are circumstances under which we can have mathematical justification, despite a possibility of being wrong.

This paper touches on many cases and questions that will reappear later across the Blueprint, such as collaboration, testimony, computer proofs, and diagrams.

DISCUSSION QUESTIONS:

1. People often talk of proofs giving an unusual sense of certainty in what they prove, that it can be no other way. Can this be reconciled with De Toffoli's fallibilist account?
2. Do you think proofs should be shareable?
3. De Toffoli says an argument that is convincing for aliens might not be shareable with humans. How do you think alien proofs might be different from human ones?
4. Is having a simil-proof enough to justify a mathematical belief?
5. Will what counts as a proof change over time? What about what counts as a simil-proof?

MÜLLER-HILL, EVA. *FORMALIZABILITY AND KNOWLEDGE ASCRIPTIONS IN MATHEMATICAL PRACTICE*

2009, *Philosophia Scientiæ. Travaux d'histoire et de philosophie des sciences,* (13-2): 21-43. **Difficulty:** Intermediate

ABSTRACT:

We investigate the truth conditions of knowledge ascriptions for the case of mathematical knowledge. The availability of a formalizable mathematical proof appears to be a natural criterion:

(*) X knows that p is true iff X has available a formalizable proof of p.

Yet, formalizability plays no major role in actual mathematical practice. We present results of an empirical study, which suggest that certain readings of (*) are not necessarily employed by mathematicians when ascribing knowledge. Further, we argue that the concept of mathematical knowledge underlying the actual use of "to know" in mathematical practice is compatible with certain philosophical intuitions, but seems to differ from philosophical knowledge conceptions underlying (*).

COMMENT:

Müller-Hill is interested in the question of when mathematicians have mathematical knowledge and to what extent it relies on the formalisability of proofs. In this paper, she undertakes an empirical investigation of mathematicians' views of when mathematicians know a theorem is true. Amazingly, while they say that they believe proofs have an exact definition and that the standards of knowledge are invariant, when presented with various toy scenarios, their judgements seem to suggest systematic context-sensitivity of a number of factors.

DISCUSSION QUESTIONS:

6. Why do you think mathematicians might say they believe one thing, while applying different standards in practice?
7. How surprising are the findings?
8. What do Müller-Hill's results mean for the nature of mathematical knowledge?

3. Testimony and Mathematics 1

ANDERSEN, LINE EDSLEV, HANNE ANDERSEN, AND HENRIK KRAGH SØRENSEN. *THE ROLE OF TESTIMONY IN MATHEMATICS*

2021, *Synthese*, 199(1): 859-870. **Difficulty:** Intermediate

ABSTRACT:

Mathematicians appear to have quite high standards for when they will rely on testimony. Many mathematicians require that a number of experts testify that they have checked the proof of a result p before they will rely on p in their own proofs without checking the proof of p. We examine why this is. We argue that for each expert who testifies that she has checked the proof of p and found no errors, the likelihood that the proof contains no substantial errors increases because different experts will validate the proof in different ways depending on their background knowledge and individual preferences. If this is correct, there is much to be gained for a mathematician from requiring that a number of experts have checked the proof of p before she will rely on p in her own proofs without checking the proof of p. In this way a mathematician can protect her own work and the work of others from errors. Our argument thus provides an explanation for mathematicians' attitude towards relying on testimony.

COMMENT:

The orthodox picture of mathematical knowledge is so individualistic that it often leaves out the mathematician themselves. In this piece, Andersen et al. look at what role testimony plays in mathematical knowledge. They thereby emphasise social features of mathematical proofs, and why this can play an important role in deciding which results to trust in the maths literature.

DISCUSSION QUESTIONS:

1. What are some of the ways that expertise is important in mathematics? What might it mean to be an expert mathematician?
2. How important is it for maths papers to go through peer review?
3. Should mathematicians rely on testimony? When is it acceptable to do so?
4. Should mathematicians be epistemically autonomous? Under what circumstances?
5. What reasons can you think of that someone might claim to have checked a proof when they actually haven't?

INGLIS, MATTHEW, ET AL. *ON MATHEMATICIANS' DIFFERENT STANDARDS WHEN EVALUATING ELEMENTARY PROOFS*

2013, *Topics in cognitive science*, 5(2): 270-282. **Difficulty:** Intermediate

ABSTRACT:

In this article, we report a study in which 109 research-active mathematicians were asked to judge the validity of a purported proof in undergraduate calculus. Significant results from our study were as follows: (a) there was substantial disagreement among mathematicians regarding whether the argument was a valid proof, (b) applied mathematicians were more likely than pure mathematicians to judge the argument valid, (c) participants who judged the argument invalid were more confident in their judgments than those who judged it valid, and (d) participants who judged the argument valid usually did not change their judgment when presented with a reason raised by other mathematicians for why the proof should be judged invalid. These findings suggest that, contrary to some claims in the literature, there is not a single standard of validity among contemporary mathematicians.

COMMENT:

In this paper, Inglis et al. carry out an empirical study to see whether mathematicians will agree in their judgements of validity. The surprising finding is that they might not, and that this cannot be explained by some simply being better at detecting errors: there seem to be substantial disagreements about what counts as a valid inference.

DISCUSSION QUESTIONS:

6. Do mathematicians have a special level of agreement?
7. These results do not fit well with the general view that a piece of reasoning is either a proof or it isn't. What do these results mean for the nature of proof?

4. Testimony and Mathematics 2

EASWARAN, KENNY. *REBUTTING AND UNDERCUTTING IN MATHEMATICS*

2015, *Philosophical Perspectives*, 29(1): 146-162. **Difficulty:** Intermediate-Advanced

ABSTRACT:

In my (2009) I argued that a central component of mathematical practice is that published proofs must be "transferable" — that is, they must be such that the author's reasons for believing the conclusion are shared directly with the reader, rather than requiring the reader to essentially rely on testimony. The goal of this paper is to explain this requirement of transferability in terms of a more general norm on defeat in mathematical reasoning that I will call "convertibility". I begin by discussing two types of epistemic defeat: "rebutting" and "undercutting". I give examples of both of these kinds of defeat from the history of mathematics. I then argue that an important requirement in mathematics is that published proofs be detailed enough to allow the conversion of rebutting defeat into undercutting defeat. Finally, I show how this sort of convertibility explains the requirement of transferability, and contributes to the way mathematics develops by the pattern referred to by Lakatos (1976) as "lemma incorporation".

COMMENT:

Easwaran brings the notions of undercutting and rebutting from epistemology to bare on the mathematical realm. These serve as motivation for conditions on proofs that Easwaran calls "transferability" and "convertibility". He argues that proposed proofs should be convertible, so that if one finds a counterexample, one can also figure out where the proof went wrong. This paper is rich with examples, though if the mathematics is too tricky for the reader one can skim over it without losing too much.

DISCUSSION QUESTIONS:

1. Easwaran discusses mathematical discovery, from students solving homework questions to mathematicians working on open problems, as a process of defeasible reasoning. Can we ever get certainty from mathematics on this picture?
2. Should proofs be transferable? Should they be convertible? What reasons might there be to reject this?
3. Easwaran links his notion of transferability to the intellectual virtue of epistemic autonomy (like Andersen et al. did above). What other intellectual virtues might it link to?
4. If convertibility is incompatible with relying on testimony in mathematics, is one of them more important than the other? Which would you rather give up?
5. In what ways can mistaken proofs still be valuable?

ANDERSEN, LINE EDSLEV, MIKKEL WILLUM JOHANSEN AND HENRIK KRAGH SØRENSEN. *MATHEMATICIANS WRITING FOR MATHEMATICIANS*

2021, *Synthese*, 198(26): 6233-6250. **Difficulty:** Easy-Intermediate

ABSTRACT:

We present a case study of how mathematicians write for mathematicians. We have conducted interviews with two research mathematicians, the talented PhD student Adam and his experienced supervisor Thomas, about a research paper they wrote together. Over the course of 2 years, Adam and Thomas revised Adam's very detailed first draft. At the beginning of this collaboration, Adam was very knowledgeable about the subject of the paper and had good presentational skills but, as a new PhD student, did not yet have experience writing research papers for mathematicians. Thus, one main purpose of revising the paper was to make it take into account the intended audience. For this reason, the changes made to the initial draft and the authors' purpose in making them provide a window for viewing how mathematicians write for mathematicians. We examined how their paper attracts the interest of the reader and prepares their proofs for validation by the reader. Among other findings, we found that their paper prepares the proofs for two types of validation that the reader can easily switch between.

COMMENT:

In this paper, Andersen et al. track the genesis of a maths research paper written in collaboration between a PhD student and his supervisor. They track changes made to sequential drafts and interview the two authors about the motivations for them, and show how the edits are designed to engage the reader in a mathematical narrative on one level, and prepare the paper for different types of validation on another level.

DISCUSSION QUESTIONS:

6. What are the two levels that a mathematical article is arguing at? How are they related?
7. How much does of the writing process described by Andersen et al. tracks making the paper's proofs more transferable in Easwaran's sense?
8. To what extent should telling a coherent story about the mathematics affect how it is validated?
9. What does the collaboration between supervisor and student tell us about mathematical collaboration?
10. What does the way a paper is best written tell us about how mathematicians pass knowledge to one another?

SCHATTSCHNEIDER, DORIS. *MARJORIE RICE (16 FEBRUARY 1923–2 JULY 2017)*

2018, *Journal of Mathematics and the Arts*, 12(1): 51-54. **Difficulty:** Easy

ABSTRACT:

Marjorie Jeuck Rice, a most unlikely mathematician, died on 2 July 2017 at the age of 94. She was born on 16 February 1923 in St. Petersburg, Florida, and raised on a tiny farm near Roseburg in southern Oregon. There she attended a one-room country school, and there her scientific interests were awakened and nourished by two excellent teachers who recognized her talent. She later wrote, 'Arithmetic was easy and I liked to discover the reasons behind the methods we used.... I was interested in the colors, patterns, and designs of nature and dreamed of becoming an artist'?

COMMENT:

Easwaran discusses the case of Marjorie Rice, an amateur mathematician who discovered new pentagon tilings. This obituary gives some details of her life and the discovery.

DISCUSSION QUESTIONS:

11. It is fairly unusual for an amateur to make important discoveries in maths. How could it be made more open to this kind of contribution? Should it?

5. The Gender Gap in Mathematics

BARROW-GREEN, JUNE. *HISTORICAL CONTEXT OF THE GENDER GAP IN MATHEMATICS*

2019, in *World Women in Mathematics 2018: Proceedings of the First World Meet. Women in Mathematics*, Carolina Araujo et al. (eds.). Springer, Cham. **Difficulty:** Easy

ABSTRACT:

This chapter is based on the talk that I gave in August 2018 at the ICM in Rio de Janeiro at the panel on The Gender Gap in Mathematical and Natural Sciences from a Historical Perspective. It provides some examples of the challenges and prejudices faced by women mathematicians during last two hundred and fifty years. I make no claim for completeness but hope that the examples will help to shed light on some of the problems many women mathematicians still face today.

COMMENT:

Barrow-Green is a historian of mathematics. In this paper she documents some of the challenges that women faced in mathematics over the last 250 years, discussing many famous women mathematicians and the prejudices and injustices they faced.

DISCUSSION QUESTIONS:

1. What social mechanisms were used to exclude women from professional mathematical practices?
2. One common theme is that the work of women mathematicians has been obscured to the historical record in various ways. How do you think this perpetuates stereotypes today?
3. To what extent were supposedly objective judgements of mathematics used to make biased assessments of women's work?
4. Why is it valuable to research the history of mathematics?

MIHALJEVIĆ, HELENA AND LUCÍA SANTAMARÍA. *AUTHORSHIP IN TOP-RANKED MATHEMATICAL AND PHYSICAL JOURNALS: ROLE OF GENDER ON SELF-PERCEPTIONS AND BIBLIOGRAPHIC EVIDENCE*

2020, *Quantitative Science Studies*, 1(4): 1468-1492. **Difficulty:** Advanced

Fragment: Introduction, pp1468-1471, and Section 4, pp1487-1489.

ABSTRACT:

Despite increasing rates of women researching in math-intensive fields, publications by female authors remain underrepresented. By analyzing millions of records from the dedicated bibliographic databases zbMATH, arXiv, and ADS, we unveil the chronological evolution of authorships by women in mathematics, physics, and astronomy. We observe a pronounced shortage of female authors in top-ranked journals, with quasistagnant figures in various distinguished periodicals in the first two disciplines and a significantly more equitable situation in the latter. Additionally, we provide an interactive open-access web interface to further examine the data. To address whether female scholars submit fewer articles for publication to relevant journals or whether they are consciously or unconsciously disadvantaged by the peer review system, we also study authors' perceptions of their submission practices and analyze around 10,000 responses, collected as part of a recent global survey of scientists. Our analysis indicates that men and women perceive their submission practices to be similar, with no evidence that a significantly lower number of submissions by women is responsible for their underrepresentation in top-ranked journals. According to the self-reported responses, a larger number of articles submitted to prestigious venues correlates rather with aspects associated with pronounced research activity, a well-established network, and academic seniority.

COMMENT:

Mihaljević and Santamaría here use large-scale quantitative research methods to investigate the gender gap in contemporary mathematics. I've recommended reading the introduction and conclusion in order to see what they were doing and what they found out, but the rest of the paper is worth looking at if you want more detailed methods and results.

DISCUSSION QUESTIONS:

5. How has the gender gap in mathematics continued in present day mathematics?
6. How objective is mathematical peer review?

6. Computer Proofs

SECCO, GISELE DALVA AND LUIZ CARLOS PEREIRA. *PROOFS VERSUS EXPERIMENTS: WITTGENSTEINIAN THEMES SURROUNDING THE FOUR-COLOR THEOREM*

2017, in *How Colours Matter to Philosophy*, Marcos Silva (ed.). Springer, Cham.

Difficulty: Intermediate-Advanced

ABSTRACT:

The Four-Colour Theorem (4CT) proof, presented to the mathematical community in a pair of papers by Appel and Haken in the late 1970's, provoked a series of philosophical debates. Many conceptual points of these disputes still require some elucidation. After a brief presentation of the main ideas of Appel and Haken's procedure for the proof and a reconstruction of Thomas Tymoczko's argument for the novelty of 4CT's proof, we shall formulate some questions regarding the connections between the points raised by Tymoczko and some Wittgensteinian topics in the philosophy of mathematics such as the importance of the surveyability as a criterion for distinguishing mathematical proofs from empirical experiments. Our aim is to show that the "characteristic Wittgensteinian invention" (Mühlhölzer 2006) – the strong distinction between proofs and experiments – can shed some light in the conceptual confusions surrounding the Four-Colour Theorem.

COMMENT:

Secco and Pereira discuss the famous proof of the Four Colour Theorem, which involved the essential use of a computer to check a huge number of combinations. They look at whether this constitutes a real proof or whether it is more akin to a mathematical experiment, a distinction that they draw from Wittgenstein.

DISCUSSION QUESTIONS:

1. Does the 4CT represent a significant change to mathematical practice?
2. Does a computer proof like that of the 4CT lack certain virtues that we would want from a proof?
3. Can mathematics have empirical elements? Should maths use experiments?
4. Are computer proofs more or less fallible than human proofs?
5. Returning to the questions of week 1 above about the relation between philosophy and mathematics, who gets to decide whether the computer proof of the 4CT is properly part of mathematics?
6. How does surveyability and the "easy reproduction of a proof" relate to the notions of shareability, transferability and convertibility seen in previous readings?

DICK, STEPHANIE. *AFTERMATH: THE WORK OF PROOF IN THE AGE OF HUMAN–MACHINE COLLABORATION*

2011, *Isis*, 102(3): 494-505. **Difficulty:** Intermediate

ABSTRACT:

During the 1970s and 1980s, a team of Automated Theorem Proving researchers at the Argonne National Laboratory near Chicago developed the Automated Reasoning Assistant, or AURA, to assist human users in the search for mathematical proofs. The resulting hybrid humans+AURA system developed the capacity to make novel contributions to pure mathematics by very untraditional means. This essay traces how these unconventional contributions were made and made possible through negotiations between the humans and the AURA at Argonne and the transformation in mathematical intuition they produced. At play in these negotiations were experimental practices, nonhumans, and nonmathematical modes of knowing. This story invites an earnest engagement between historians of mathematics and scholars in the history of science and science studies interested in experimental practice, material culture, and the roles of nonhumans in knowledge making.

COMMENT:

Dick traces the history of the AURA automated reasoning assistant in the 1970s and 80s, arguing that the introduction of the computer system led to novel contributions to mathematics by unconventional means. Dick's emphasis is on the AURA system as changing the material culture of mathematics, and thereby leading to collaboration and even negotiations between the mathematicians and the computer system.

DISCUSSION QUESTIONS:

7. How can collaborating with a computer affect how one does mathematics?
8. Is working with a computer different to collaborating with another human mathematician? Will this change what the "negotiations" are?

7. Diagrammatic Proofs 1

DE TOFFOLI, SILVIA AND VALERIA GIARDINO. *AN INQUIRY INTO THE PRACTICE OF PROVING IN LOW-DIMENSIONAL TOPOLOGY*

2015, in *From Logic to Practice*, Gabriele Lolli, Giorgio Venturi and Marco Panza (eds.). Springer International Publishing.

Difficulty: Intermediate-Advanced

ABSTRACT:

The aim of this article is to investigate specific aspects connected with visualization in the practice of a mathematical subfield: low-dimensional topology. Through a case study, it will be established that visualization can play an epistemic role. The background assumption is that the consideration of the actual practice of mathematics is relevant to address epistemological issues. It will be shown that in low-dimensional topology, justifications can be based on sequences of pictures. Three theses will be defended. First, the representations used in the practice are an integral part of the mathematical reasoning. As a matter of fact, they convey in a material form the relevant transitions and thus allow experts to draw inferential connections. Second, in low-dimensional topology experts exploit a particular type of manipulative imagination which is connected to intuition of two- and three-dimensional space and motor agency. This imagination allows recognizing the transformations which connect different pictures in an argument. Third, the epistemic—and inferential—actions performed are permissible only within a specific practice: this form of reasoning is subject-matter dependent. Local criteria of validity are established to assure the soundness of representationally heterogeneous arguments in low-dimensional topology.

COMMENT:

De Toffoli and Giardino look at proof practices in low-dimensional topology, and especially a proof by Rolfsen that relies on epistemic actions on a diagrammatic representation. They make the case that the many diagrams are used to trigger our manipulative imagination to make inferential moves which cannot be reduced to formal statements without loss of intuition.

DISCUSSION QUESTIONS:

1. Many traditional approaches to proof rule out diagrams as an extraneous part of proofs that cannot play an essential role. How well does that stand up to De Toffoli & Giardino's case study?
2. What is the role of manipulative imagination in mathematical reasoning in topology?
3. How much do you think being able to "see" topological transformations depends on being an experienced topologist? Does intuition have to be trained?
4. What is the relationship between a normal topology proof and a formalisation of it? What does a formalisation capture? What might it miss?
5. Is a subcommunity of mathematics free to choose any criteria of validity they like for proofs?

MCCALLUM, KATE. *UNTANGLING KNOTS: EMBODIED DIAGRAMMING PRACTICES IN KNOT THEORY*

2019, *Journal of Humanistic Mathematics*, 9(1): 178-199. **Difficulty:** Easy-Intermediate

ABSTRACT:

The low visibility and specialised languages of mathematical work pose challenges for the ethnographic study of communication in mathematics, but observation-based study can offer a real-world grounding to questions about the nature of its methods. This paper uses theoretical ideas from linguistic pragmatics to examine how mutual understandings of diagrams are achieved in the course of conference presentations. Presenters use shared knowledge to train others to interpret diagrams in the ways favoured by the community of experts, directing an audience's attention so as to develop a shared understanding of a diagram's features and possible manipulations. In this way, expectations about the intentions of others and appeals to knowledge about the manipulation of objects play a part in the development and communication of concepts in mathematical discourse.

COMMENT:

McCallum is an ethnographer and artist, who in this piece explores the way in which mathematicians use diagrams in conference presentations, especially in knot theory. She emphasises that there are a large number of ways that diagrams can facilitate communication and understanding. The diagrams are dynamic in many way, and she shows how the way in which a speaker interacts with the diagram (through drawing, erasing, labelling, positioning, emphasising etc.) is part of explaining the mathematics it represents.

DISCUSSION QUESTIONS:

6. How might the active presentation of a diagram aid the audience's manipulative imagination?
7. How important are the physical materials of mathematics?
8. I was once subjected to a training day in which a Pro-Dean of Research declared they wanted to remove all blackboards from the maths department. Would this make a difference to the mathematical practices? What about to the mathematics produced?

8. Diagrammatic Proofs 2

CARTER, JESSICA. *DIAGRAMS AND PROOFS IN ANALYSIS*

2010, *International Studies in the Philosophy of Science*, 24(1): 1-14. **Difficulty:** Intermediate-Advanced

ABSTRACT:

This article discusses the role of diagrams in mathematical reasoning in the light of a case study in analysis. In the example presented certain combinatorial expressions were first found by using diagrams. In the published proofs the pictures were replaced by reasoning about permutation groups. This article argues that, even though the diagrams are not present in the published papers, they still play a role in the formulation of the proofs. It is shown that they play a role in concept formation as well as representations of proofs. In addition we note that 'visualization' is used in two different ways. In the first sense 'visualization' denotes our inner mental pictures, which enable us to see that a certain fact holds, whereas in the other sense 'visualization' denotes a diagram or representation of something.

COMMENT:

In this paper, Carter discusses a case study from free probability theory in which diagrams were used to inspire definitions and proof strategies. Interestingly, the diagrams were not present in the published results making them dispensable in one sense, but Carter argues that they are essential in the sense that their discovery relied on the visualisation supplied by the diagrams.

DISCUSSION QUESTIONS:

1. Do you think it is important that diagrams are dispensable in a mathematical proof?
2. What are the two senses of visualisation that Carter discusses? Are the two related?
3. In what sense are the diagrams Carter considers essential to the discovery of proofs and definitions?
4. How do you think a mathematician might read a paper in which the diagrams have been omitted? Would they reconstruct them to gain understanding?
5. Compare Carter's claims with those of De Toffoli & Giardino before. In one case the focus is on the context of discovery, while the other is on the context of justification. How separate are these contexts? Are the claims in these papers in tension?

FRANCOIS, KAREN AND ERIC VANDENDRIESSCHE. *REASSEMBLING MATHEMATICAL PRACTICES: A PHILOSOPHICAL-ANTHROPOLOGICAL APPROACH*

2016, *Revista Latinoamericana de Etnomatemática Perspectivas Socioculturales de la Educación Matemática*, 9(2): 144-167.

Difficulty: Easy-Intermediate

ABSTRACT:

In this paper we first explore how Wittgenstein's philosophy provides a conceptual tools to discuss the possibility of the simultaneous existence of culturally different mathematical practices. We will argue that Wittgenstein's later work will be a fruitful framework to serve as a philosophical background to investigate ethnomathematics (Wittgenstein 1973). We will give an overview of Wittgenstein's later work which is referred to by many researchers in the field of ethnomathematics. The central philosophical investigation concerns Wittgenstein's shift to abandoning the essentialist concept of language and therefore denying the existence of a universal language. Languages—or 'language games' as Wittgenstein calls them—are immersed in a form of life, in a cultural or social formation and are embedded in the totality of communal activities. This gives rise to the idea of rationality as an invention or as a construct that emerges in specific local contexts. In the second part of the paper we introduce, analyse and compare the mathematical aspects of two activities known as string figure-making and sand drawing, to illustrate Wittgenstein's ideas. Based on an ethnomathematical comparative analysis, we will argue that there is evidence of invariant and distinguishing features of a mathematical rationality, as expressed in both string figure-making and sand drawing practices, from one society to another. Finally, we suggest that a philosophical-anthropological approach to mathematical practices may allow us to better understand the interrelations between mathematics and cultures. Philosophical investigations may help the reflection on the possibility of culturally determined ethnomathematics, while an anthropological approach, using ethnographical methods, may afford new materials for the analysis of ethnomathematics and its links to the cultural context. This combined approach will help us to better characterize mathematical practices in both sociological and epistemological terms.

COMMENT:

Francois and Vandendriessche here present a later Wittgensteinian approach to "ethnomathematics": mathematics practiced outside of mainstream Western contexts, often focused on indigenous or tribal groups. They focus on two case studies, string-figure making and sand-drawing, in different geographic and cultural contexts, looking at how these practices are mathematical.

DISCUSSION QUESTIONS:

6. What makes a practice like string-figure making or sand-drawing mathematical?
7. What is the relationship between mathematics and culture?
8. How are the sand-drawing practices similar to Carter's diagram case studies? How are they different?

9. Online Mathematics

MARTIN, URSULA AND ALISON PEASE. *MATHEMATICAL PRACTICE, CROWDSOURCING, AND SOCIAL MACHINES*

2013, in *Intelligent Computer Mathematics. CICM 2013. Lecture Notes in Computer Sciences*, Carette, J. et al. (eds.). Springer.

Difficulty: Intermediate

ABSTRACT:

The highest level of mathematics has traditionally been seen as a solitary endeavour, to produce a proof for review and acceptance by research peers. Mathematics is now at a remarkable inflexion point, with new technology radically extending the power and limits of individuals. Crowdsourcing pulls together diverse experts to solve problems; symbolic computation tackles huge routine calculations; and computers check proofs too long and complicated for humans to comprehend.

The Study of Mathematical Practice is an emerging interdisciplinary field which draws on philosophy and social science to understand how mathematics is produced. Online mathematical activity provides a novel and rich source of data for empirical investigation of mathematical practice - for example the community question-answering system mathoverflow contains around 40,000 mathematical conversations, and polymath collaborations provide transcripts of the process of discovering proofs. Our preliminary investigations have demonstrated the importance of "soft" aspects such as analogy and creativity, alongside deduction and proof, in the production of mathematics, and have given us new ways to think about the roles of people and machines in creating new mathematical knowledge. We discuss further investigation of these resources and what it might reveal.

Crowdsourced mathematical activity is an example of a "social machine", a new paradigm, identified by Berners-Lee, for viewing a combination of people and computers as a single problem-solving entity, and the subject of major international research endeavours. We outline a future research agenda for mathematics social machines, a combination of people, computers, and mathematical archives to create and apply mathematics, with the potential to change the way people do mathematics, and to transform the reach, pace, and impact of mathematics research.

COMMENT:

In this paper, Martin and Pease look at how mathematics happens online, emphasising how this embodies the picture of mathematics given by Polya and Lakatos, two central figures in philosophy of mathematical practice. They look at multiple venues of online mathematics, including the polymath projects of collaborative problem-solving, and mathoverflow, which is a question-and-answer forum. By looking at the discussions that take place when people are doing maths online, they argue that you can get rich new kinds of data about the processes of mathematical discovery and understanding. They discuss how online mathematics can become a "social machine", and how this can open up new ways of doing mathematics.

DISCUSSION QUESTIONS:

1. Is "massively" collaborative mathematics possible?
2. In their analysis of the mini-polymath, Martin & Pease found a large number of examples being used. What is the role of examples in coming to understand a problem?
3. Are collaborative proofs more reliable?
4. Do you think online mathematics leads to the emergence of its own mathematical culture?
5. Is online mathematics a social machine? Has research mathematics always been a social machine, or is this a radical change in mathematics?
6. Can a social machine "think like a mathematician"? Can it do even better

MELFI, THEODORE. *HIDDEN FIGURES*

2016, [Feature film], 20th Century Fox. **Difficulty:** Easy

ABSTRACT:

The story of a team of female African-American mathematicians who served a vital role in NASA during the early years of the U.S. space program.

COMMENT:

This film depicts a historical biopic of African American female mathematicians working at NASA in the 1960s, focusing on the story of Katherine Johnson. In it, the plot depicts struggles with racism and sexism, as well as the impacts of the move from human calculation to the use of computers.

10. Enormous Proofs

STEINGART, ALMA. *A GROUP THEORY OF GROUP THEORY: COLLABORATIVE MATHEMATICS AND THE 'UNINVENTION' OF A 1000-PAGE PROOF*

2012, *Social Studies of Science*, 42(2): 185-213. **Difficulty:** Intermediate-Advanced

ABSTRACT:

Over a period of more than 30 years, more than 100 mathematicians worked on a project to classify mathematical objects known as finite simple groups. The Classification, when officially declared completed in 1981, ranged between 300 and 500 articles and ran somewhere between 5,000 and 10,000 journal pages. Mathematicians have hailed the project as one of the greatest mathematical achievements of the 20th century, and it surpasses, both in scale and scope, any other mathematical proof of the 20th century. The history of the Classification points to the importance of face-to-face interaction and close teaching relationships in the production and transformation of theoretical knowledge. The techniques and methods that governed much of the work in finite simple group theory circulated via personal, often informal, communication, rather than in published proofs. Consequently, the printed proofs that would constitute the Classification Theorem functioned as a sort of shorthand for and formalization of proofs that had already been established during personal interactions among mathematicians. The proof of the Classification was at once both a material artifact and a crystallization of one community's shared practices, values, histories, and expertise. However, beginning in the 1980s, the original proof of the Classification faced the threat of 'uninvention'. The papers that constituted it could still be found scattered throughout the mathematical literature, but no one other than the dwindling community of group theorists would know how to find them or how to piece them together. Faced with this problem, finite group theorists resolved to produce a 'second-generation proof' to streamline and centralize the Classification. This project highlights that the proof and the community of finite simple groups theorists who produced it were co-constitutive–one formed and reformed by the other.

COMMENT:

Steingart is a sociologist who charts the history and sociology of the development of the extremely large and highly collaborative Classification Theorem. She shows that the proof involved a community deciding on shared values, standards of reliability, expertise, and ways of communicating. For example, the community became tolerant of so-called "local errors" so long as these did not put the main result at risk. Furthermore, Steingart discusses how the proof's text is distributed across a wide number of places and requires expertise to navigate, leaving the proof in danger of uninvention if the experts retire from mathematics.

DISCUSSION QUESTIONS:

1. Does it challenge the traditional conception of mathematical knowledge if no mathematician individually knows all of the pieces of the proof of the Classification Theorem?
2. Steingart claims that the circulation of knowledge and adjudication cannot be separated. Is this a necessary feature of mathematical knowledge, or is it a problem for its reliability? Or both/neither?
3. Does this case make us rethink the role of testimony in mathematics?
4. What does the danger of the theorem being "uninvented" mean for the idea that mathematical knowledge is cumulative and eternal?
5. Should the group theorists really be confident that there are only fixable, local errors in the proof, and not a more major error?

HABGOOD-COOTE, JOSHUA AND FENNER TANSWELL. *GROUP KNOWLEDGE AND MATHEMATICAL COLLABORATION: A PHILOSOPHICAL EXAMINATION OF THE CLASSIFICATION OF FINITE SIMPLE GROUPS*

2021, *Episteme*, pp.1-27. doi:10.1017/epi.2021.26. **Difficulty:** Intermediate-Advanced

ABSTRACT:

In this paper we apply social epistemology to mathematical proofs and their role in mathematical knowledge. The most famous modern collaborative mathematical proof effort is the Classification of Finite Simple Groups. The history and sociology of this proof have been well-documented by Alma Steingart (2012), who highlights a number of surprising and unusual features of this collaborative endeavour that set it apart from smaller-scale pieces of mathematics. These features raise a number of interesting philosophical issues, but have received very little attention. In this paper, we will consider the philosophical tensions that Steingart uncovers, and use them to argue that the best account of the epistemic status of the Classification Theorem will be essentially and ineliminably social. This forms part of the broader argument that in order to understand mathematical proofs, we must appreciate their social aspects.

COMMENT:

In this paper, we take on some of the philosophical issues raised by Steingart's case study. We look at how notions of proof and justification need to be understood as social in order to apply to the practices of the group theory community. We draw on recent work in social epistemology to try to explain some of the otherwise surprising standards of the mathematicians, such as by using the concept of "coverage-supported justification" to explain how mathematicians may be justified in believing there are no major errors in their work.

DISCUSSION QUESTIONS:

6. Is it okay for proofs to contain errors, so long as they are "fixable"?
7. What does it mean to "know a proof"?
8. Who knows the proof of the classification theorem?
9. Should the group theorists really be confident there are no more finite simple groups they've missed?

11. Proofs as Dialogues

DUTILH NOVAES, CATARINA. *THE DIALOGICAL ROOTS OF DEDUCTION: HISTORICAL, COGNITIVE, AND PHILOSOPHICAL PERSPECTIVES ON REASONING*

2020, Cambridge University Press. **Difficulty:** Intermediate

Fragment: Chapter 11, "A Dialogical Account of Proofs in Mathematical Practice"

ABSTRACT:

This comprehensive account of the concept and practices of deduction is the first to bring together perspectives from philosophy, history, psychology and cognitive science, and mathematical practice. Catarina Dutilh Novaes draws on all of these perspectives to argue for an overarching conceptualization of deduction as a dialogical practice: deduction has dialogical roots, and these dialogical roots are still largely present both in theories and in practices of deduction. Dutilh Novaes' account also highlights the deeply human and in fact social nature of deduction, as embedded in actual human practices; as such, it presents a highly innovative account of deduction. The book will be of interest to a wide range of readers, from advanced students to senior scholars, and from philosophers to mathematicians and cognitive scientists.

COMMENT:

This book by Dutilh Novaes recently won the coveted Lakatos Award. In it, she develops a dialogical account of deduction, where she argues that deduction is implicitly dialogical. Proofs represent dialogues between Prover, who is aiming to establish the theorem, and Skeptic, who is trying to block the theorem. However, the dialogue is both partially adversarial (the two characters have opposite goals) and partially cooperative: the Skeptic's objections make sure that the Prover must make their proof clear, convincing, and correct. In this chapter, Dutilh Novaes applies her model to mathematical practice, and looks at the way social features of maths embody the Prover-Skeptic dialogical model.

DISCUSSION QUESTIONS:

1. What is the difference between a proof and a proof presentation?
2. Is the peer review process like a dialogue between author and referee? In what ways might it be different?
3. Is mathematics a collaboratively adversarial enterprise?
4. One trouble with the controversy about Mochizuki's proposed proof of the abc conjecture is the disagreement over who counts as a relevant expert. Who do you think should count?
5. Dutilh Novaes lists a number of different functions of proofs. How well do the various unusual proofs (e.g. probabilistic, computer, diagrammatic, collaborative etc.) we have seen in previous weeks match the different functions?

MORRIS, REBECCA LEA. *INTELLECTUAL GENEROSITY AND THE REWARD STRUCTURE OF MATHEMATICS*

2021, *Synthese*, 199(1): 345-367. **Difficulty:** Intermediate

ABSTRACT:

Prominent mathematician William Thurston was praised by other mathematicians for his intellectual generosity. But what does it mean to say Thurston was intellectually generous? And is being intellectually generous beneficial? To answer these questions I turn to virtue epistemology and, in particular, Roberts and Wood's (2007) analysis of intellectual generosity. By appealing to Thurston's own writings and interviewing mathematicians who knew and worked with him, I argue that Roberts and Wood's analysis nicely captures the sense in which he was intellectually generous. I then argue that intellectual generosity is beneficial because it counteracts negative effects of the reward structure of mathematics that can stymie mathematical progress.

COMMENT:

In this paper, Morris looks at ascriptions of intellectual generosity in mathematics, focusing on the mathematician William Thurston. She looks at how generosity should be characterised, and argues that it is beneficial in counteract some of the negative effects of the reward structure of mathematics.

DISCUSSION QUESTIONS:

6. What does it mean to be intellectually generous?
7. Does being generous make you a better mathematician?
8. What is the relationship between the intellectual virtues of individuals and the state of a subfield of mathematics?
9. Are theorem-credits a good reward system for maths?
10. Will the priority rule always make sure the first person to prove something gets the credit? In what ways might this go wrong?